

# Gröbner-Shirshov bases for $L$ -algebras\*

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**Abstract:** In this paper, we firstly establish Composition-Diamond lemma for  $\Omega$ -algebras. We give a Gröbner-Shirshov basis of the free  $L$ -algebra as a quotient algebra of a free  $\Omega$ -algebra, and then the normal form of the free  $L$ -algebra is obtained. We secondly establish Composition-Diamond lemma for  $L$ -algebras. As applications, we give Gröbner-Shirshov bases of the free dialgebra and the free product of two  $L$ -algebras, and then we show four embedding theorems of  $L$ -algebras: 1) Every countably generated  $L$ -algebra can be embedded into a two-generated  $L$ -algebra. 2) Every  $L$ -algebra can be embedded into a simple  $L$ -algebra. 3) Every countably generated  $L$ -algebra over a countable field can be embedded into a simple two-generated  $L$ -algebra. 4) Three arbitrary  $L$ -algebras  $A$ ,  $B$ ,  $C$  over a field  $k$  can be embedded into a simple  $L$ -algebra generated by  $B$  and  $C$  if  $|k| \leq \dim(B * C)$  and  $|A| \leq |B * C|$ , where  $B * C$  is the free product of  $B$  and  $C$ .

**Key words:** Gröbner-Shirshov basis;  $\Omega$ -algebra; dialgebra;  $L$ -algebra.

**AMS 2000 Subject Classification:** 16S15, 13P10, 16W99, 17D99

## 1 Introduction

The theories of Gröbner-Shirshov bases and Gröbner bases were invented independently by A.I. Shirshov ([11], 1962) for non-commutative and non-associative algebras, and by

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\*Supported by the NNSF of China (Nos.10771077, 10911120389) and the NSF of Guangdong Province (No.06025062).

<sup>†</sup>Supported by RFBR 01-09-00157, LSS-344.2008.1 and SB RAS Integration grant No. 2009.97 (Russia).

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H. Hironaka ([5], 1964) and B. Buchberger ([3], 1965) for commutative algebras. Gröbner-Shirshov technique is proved to be very useful in the study of presentations of many kinds of algebras by generators and defining relations.

V. Drensky and R. Holtkamp [4] constructed Gröbner bases theory for algebras with multiple operations ( $\Omega$ -algebras, where  $\Omega$  consists of  $n$ -ary operations,  $n \geq 2$ ) and proved the Diamond lemma for  $\Omega$ -algebras. For associative  $\Omega$ -algebras, Gröbner-Shirshov bases and Composition-Diamond lemma were established in [1].

The class of  $L$ -algebras was invented by P. Leroux [7]. In [6], P. Leroux gave a coalgebraic framework of directed graphs equipped with weights in terms of  $L$ -coalgebras, which are  $k$ -spaces with two co-operations  $\Delta_M, \tilde{\Delta}_M$  verifying the entanglement relation:

$$(\tilde{\Delta}_M \otimes id)\Delta_M = (id \otimes \Delta_M)\tilde{\Delta}_M.$$

$L$ -algebras arise from this coding weighted directed graphs [7]. They are algebras over a field  $k$  with two operations  $\prec, \succ$  satisfied one identity:

$$(x \succ y) \prec z = x \succ (y \prec z).$$

In fact, many types of algebras are  $L$ -algebras. For example, associative algebras. In this case, the two operations coincide with the associative product.

P. Leroux found a normal form of a free  $L$ -algebra in [7].

In this paper, we establish Composition-Diamond lemma for  $\Omega$ -algebras, where  $\Omega$  consists of  $n$ -ary operations,  $n \geq 1$ . This generalizes the result in V. Drensky and R. Holtkamp [4]. As a result, we give another linear basis for the free  $L$ -algebra by using Composition-Diamond lemma for  $\Omega$ -algebras. Then we continue to study  $L$ -algebras and establish the Composition-Diamond lemma for  $L$ -algebras. As applications, we prove embedding theorems for  $L$ -algebras:

- 1) Every countably generated  $L$ -algebra over a field  $k$  can be embedded into a two-generated  $L$ -algebra.
- 2) Every  $L$ -algebra over a field  $k$  can be embedded into a simple  $L$ -algebra.
- 3) Every countably generated  $L$ -algebra over a countable field  $k$  can be embedded into a simple two-generated  $L$ -algebra.
- 4) Three arbitrary  $L$ -algebras  $A, B, C$  over a field  $k$  can be embedded into a simple  $L$ -algebra generated by  $B$  and  $C$  if  $|k| \leq \dim(B * C)$  and  $|A| \leq |B * C|$ , where  $B * C$  is the free product of  $B$  and  $C$ .

We also give the Gröbner-Shirshov bases of a free dialgebra and the free product of two  $L$ -algebras, respectively, and then the normal forms of such algebras are obtained.

## 2 Composition-Diamond lemma for $\Omega$ -algebras

In this section, we establish the Composition-Diamond lemma for  $\Omega$ -algebras.

Let  $k$  be a field,  $X$  a set of variables,  $\Omega$  a set of multilinear operations, and

$$\Omega = \cup_{n \geq 1} \Omega_n,$$

where  $\Omega_n = \{\delta_i^{(n)} | i \in I_n\}$  is the set of  $n$ -ary operations,  $n = 1, 2, \dots$ . Now, we define “ $\Omega$ -words”. (Usually they are called terms in  $X$  and  $\Omega$ , see, for example [10].)

Define

$$(X, \Omega)_0 = X.$$

For  $m \geq 1$ , define

$$(X, \Omega)_m = X \cup \Omega((X, \Omega)_{m-1})$$

where

$$\Omega((X, \Omega)_{m-1}) = \cup_{t=1}^{\infty} \{ \delta_i^{(t)}(u_1, u_2, \dots, u_t) \mid \delta_i^{(t)} \in \Omega_t, u_j \in (X, \Omega)_{m-1} \}.$$

Let

$$(X, \Omega) = \bigcup_{m=0}^{\infty} (X, \Omega)_m.$$

Then each element in  $(X, \Omega)$  is called an  $\Omega$ -word.

Let  $k(X, \Omega)$  be a  $k$ -linear space with  $k$ -basis  $(X, \Omega)$ . For any  $\delta_i^{(t)} \in \Omega_t$ , extend linearly  $\delta_i^{(t)}$  to  $k(X, \Omega)$ . Then  $k(X, \Omega)$  is a free  $\Omega$ -algebra generated by  $X$ . We call the elements of  $k(X, \Omega)$  as  $\Omega$ -polynomials.

Let  $k(X, \Omega)$  be the free  $\Omega$ -algebra defined as above and  $\star \notin X$ . By a  $\star$ - $\Omega$ -word we mean any expression in  $(X \cup \{\star\}, \Omega)$  with only one occurrence of  $\star$ . Let  $u$  be a  $\star$ - $\Omega$ -word and  $s \in k(X, \Omega)$ . Then we call  $u|_s = u|_{\star \mapsto s}$  an  $s$ - $\Omega$ -word.

Similar to  $\star$ - $\Omega$ -word, we define  $(\star_1, \star_2)$ - $\Omega$ -words as expressions in  $(X \cup \{\star_1, \star_2\}, \Omega)$  with only one occurrence of  $\star_1$  and only one occurrence of  $\star_2$ . Let  $u$  be a  $(\star_1, \star_2)$ - $\Omega$ -word and  $s_1, s_2 \in k(X, \Omega)$ . We call

$$u|_{s_1, s_2} = u|_{\star_1 \mapsto s_1, \star_2 \mapsto s_2}$$

an  $s_1$ - $s_2$ - $\Omega$ -word.

Let  $>$  be a well ordering on  $(X, \Omega)$ . Then  $>$  is called monomial if for any  $\star$ - $\Omega$ -word  $w$  and any  $u, v \in (X, \Omega)$ ,  $u > v$  implies  $w|_u > w|_v$ . The following example shows that such a monomial ordering on  $(X, \Omega)$  exists.

**Example 2.1** Suppose that  $X$  and  $\Omega$  are well-ordered sets. For any  $u \in (X, \Omega)$ , if  $u = x \in X$ , define  $wt(u) = (1, x)$ ; if  $u = \delta^{(i)}(u_1, u_2, \dots, u_i)$ , define  $wt(u) = (|u|_{\Omega} + |u|_X, |u|_X, \delta^{(i)}, u_1, u_2, \dots, u_i)$ , where  $|u|_T$  means the number of  $t \in T$  in  $u$ . For any  $u, v \in (X, \Omega)$ , we define

$$u > v \iff wt(u) > wt(v) \quad \text{lexicographically}$$

by induction on  $|u|_{\Omega} + |u|_X + |v|_{\Omega} + |v|_X$ . Then it is easy to see that  $>$  is a monomial ordering on  $(X, \Omega)$ .

**Remark:** In V. Drensky and R. Holtkamp [4], they use the ordering (2) (see next section). However, the ordering (2) is not well-ordered on  $(X, \Omega)$ . For example, let  $\delta^{(1)}, \zeta^{(1)} \in \Omega_1$  with  $\delta^{(1)} > \zeta^{(1)}$  and  $x \in X$ . Then for the ordering (2), we have an infinite descending chain

$$\delta^{(1)}(x) > \zeta^{(1)}(\delta^{(1)}(x)) > \zeta^{(1)}(\zeta^{(1)}(\delta^{(1)}(x))) > \dots$$

From now on, in this section, we assume that  $(X, \Omega)$  is equipped with a monomial ordering  $>$ .

For any  $\Omega$ -polynomial  $0 \neq f \in k(X, \Omega)$ , let  $\bar{f}$  be the leading  $\Omega$ -word of  $f$ . If the coefficient of  $\bar{f}$  is 1, then  $f$  is called monic.

**Definition 2.2** Let  $f, g$  be two monic  $\Omega$ -polynomials. If there exists an  $\Omega$ -word  $w = \bar{f} = u|_{\bar{g}}$  for some  $\star$ - $\Omega$ -word  $u$ , then we call  $(f, g)_w = f - u|_g$  the inclusion composition of  $f$  and  $g$  with respect to  $w$ . If this is the case,  $w$  is called the ambiguity of the composition  $(f, g)_w$ .

**Definition 2.3** Let  $S$  be a set of monic  $\Omega$ -polynomials. Then the composition  $(f, g)_w$  is called trivial modulo  $(S, w)$ , denoted by  $(f, g)_w \equiv 0 \pmod{(S, w)}$ , if

$$(f, g)_w = \sum \alpha_i u_i|_{s_i},$$

where each  $\alpha_i \in k$ ,  $u_i$   $\star$ - $\Omega$ -word,  $s_i \in S$  and  $u_i|_{\bar{s}_i} < w$ .

$S$  is called a Gröbner-Shirshov basis in  $k(X, \Omega)$  if for any  $f, g \in S$ ,  $(f, g)_w \equiv 0 \pmod{(S, w)}$ .

A subset  $I$  of  $k(X, \Omega)$  is called an  $\Omega$ -ideal of  $k(X, \Omega)$  if  $I$  is a subspace such that for any  $\star$ - $\Omega$ -word  $u$ ,

$$u|_I = \{u|_f | f \in I\} \subseteq I.$$

**Theorem 2.4** (Composition-Diamond lemma for  $\Omega$ -algebras) Let  $S$  be a set of monic  $\Omega$ -polynomials in  $k(X, \Omega)$ ,  $>$  a monomial ordering on  $(X, \Omega)$  and  $Id(S)$  the  $\Omega$ -ideal of  $k(X, \Omega)$  generated by  $S$ . Then the following statements are equivalent:

I )  $S$  is a Gröbner-Shirshov basis in  $k(X, \Omega)$ .

II )  $f \in Id(S) \Rightarrow \bar{f} = u|_{\bar{s}}$  where  $u|_{\bar{s}}$  is an  $s$ - $\Omega$ -word,  $s \in S$ .

III)  $Irr(S) = \{\omega \in (X, \Omega) | \omega \neq u|_{\bar{s}}, u|_{\bar{s}} \text{ is an } s\text{-}\Omega\text{-word, } s \in S\}$  is a  $k$ -basis of the algebra  $k(X, \Omega | S) = k(X, \Omega) / Id(S)$ .

**Proof.**  $I) \Rightarrow II)$ . Let  $S$  be a Gröbner-Shirshov basis in  $k(X, \Omega)$  and  $0 \neq f \in Id(S)$ . We can assume that

$$f = \sum_{i=1}^n \alpha_i u_i|_{s_i},$$

where each  $\alpha_i \in k$ ,  $s_i \in S$  and  $u_i|_{s_i}$   $s_i$ - $\Omega$ -word. Let

$$w_i = u_i|_{\bar{s}_i}, w_1 = w_2 = \dots = w_l > w_{l+1} \geq \dots$$

We prove the result by induction on  $l$  and  $w_1$ .

If  $l = 1$ , then  $\bar{f} = \overline{u_1|_{s_1}} = u_1|_{\bar{s}_1}$  and the result holds. Assume that  $l \geq 2$ . Then

$$\alpha_1 u_1|_{s_1} + \alpha_2 u_2|_{s_2} = (\alpha_1 + \alpha_2) u_1|_{s_1} + \alpha_2 (u_2|_{s_2} - u_1|_{s_1}).$$

There are two cases to consider:

1)  $\bar{s}_1$  and  $\bar{s}_2$  are disjoint in  $w_1$ . Then there exists a  $(\star_1, \star_2)$ - $\Omega$ -word  $\Pi$  such that

$$\Pi|_{\bar{s}_1, \bar{s}_2} = u_1|_{\bar{s}_1} = u_2|_{\bar{s}_2}$$

and

$$u_1|_{s_1} - u_2|_{s_2} = \Pi|_{s_1, \bar{s}_2} - \Pi|_{\bar{s}_1, s_2} = -\Pi|_{s_1, s_2 - \bar{s}_2} + \Pi|_{s_1 - \bar{s}_1, s_2}.$$

Since  $\overline{s_2 - \overline{s_2}} < \overline{s_2}$  and  $\overline{s_1 - \overline{s_1}} < \overline{s_1}$ , we have  $\overline{\Pi|_{s_1, s_2 - \overline{s_2}}} < w_1$  and  $\overline{\Pi|_{s_1 - \overline{s_1}, s_2}} < w_1$ . Noting that  $\Pi|_{s_1, s_2 - \overline{s_2}}$  is linear combination of  $s_1$ - $\Omega$ -words and  $\Pi|_{s_1 - \overline{s_1}, s_2}$  is linear combination of  $s_2$ - $\Omega$ -words, we have

$$u_1|_{s_1} \equiv u_2|_{s_2} \pmod{(S, w_1)}.$$

Thus, if  $\alpha_1 + \alpha_2 \neq 0$  or  $l > 2$ , then the result follows from the induction on  $l$ . For the case  $\alpha_1 + \alpha_2 = 0$  and  $l = 2$ , we use the induction on  $w_1$ . The result follows.

2) One of  $\overline{s_1}$ ,  $\overline{s_2}$  is contained in the other. We may assume that  $\overline{s_2}$  is contained in  $\overline{s_1}$ , i.e.,  $\overline{s_1} = u|_{\overline{s_2}}$  for some  $s_2$ - $\Omega$ -word  $u|_{s_2}$ . Thus,

$$u_1|_{s_1} - u_2|_{s_2} = u_1|_{s_1} - u_1|_{u|_{s_2}} = u_1|_{s_1 - u|_{s_2}}.$$

Since  $S$  is a Gröbner-Shirshov basis in  $k(X, \Omega)$ , we have

$$s_1 - u|_{s_2} = \sum_t \alpha_t v_t|_{s_t}$$

where each  $\alpha_t \in k$ ,  $s_t \in S$ ,  $v_t|_{s_t}$   $s_t$ - $\Omega$ -word, and  $v_t|_{\overline{s_t}} < \overline{s_1}$ . So  $u_1|_{v_t|_{\overline{s_t}}} < w_1$  for any  $t$ . Now, the result follows.

$II) \Rightarrow III)$ . For any  $f \in k(X, \Omega)$ , we may express  $f$  as

$$f = \sum_{u_i \in Irr(S), u_i \leq \overline{f}} \alpha_i u_i + \sum_{s_j \in S, u_j|_{\overline{s_j}} \leq \overline{f}} \beta_j u_j|_{s_j},$$

where  $\alpha_i, \beta_j \in k$  and  $u_j|_{s_j}$   $s_j$ - $\Omega$ -word. So any  $f \in k(X, \Omega)$  can be expressed modulo  $Id(S)$  as a linear combination of elements from  $Irr(S)$ . That is,  $Irr(S)$  spans  $k(X, \Omega|S)$  as  $k$ -space.

Suppose  $g = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$  in  $k(X, \Omega|S)$ , where  $u_i \in Irr(S)$ ,  $\alpha_i \neq 0$ ,  $i = 1, 2, \dots, n$  and  $u_1 > u_2 > \cdots > u_n$ . Then in  $k(X, \Omega)$ ,  $g \in Id(S)$ . By  $II)$ ,  $\overline{g} = u_1 = u|_{\overline{s}} \notin Irr(S)$ , a contradiction.

$III) \Rightarrow I)$ . For any composition  $(f, g)_w$  in  $S$ , by  $III)$ ,

$$(f, g)_w = \sum_{s_j \in S, u_j|_{\overline{s_j}} \leq \overline{(f, g)_w}} \beta_j u_j|_{s_j}.$$

Since  $\overline{(f, g)_w} < w$ ,  $(f, g)_w \equiv 0 \pmod{(S, w)}$ . □

**Remark:** When  $\Omega$  only contains  $n$ -ary multilinear operations for  $n \geq 2$ , the situation is just the same as that established by V. Drensky and R. Holtkamp in [4].

### 3 Gröbner-Shirshov bases for free $L$ -algebras

In this section, by using the Composition-Diamond lemma for  $\Omega$ -algebras (Theorem 2.4), we give a Gröbner-Shirshov basis of the free  $L$ -algebra and then a  $k$ -basis of such an algebra is obtained.

**Definition 3.1** [7] An  $L$ -algebra is a  $k$ -space  $L$  equipped with two binary  $k$ -linear operations  $\prec, \succ: L \otimes L \rightarrow L$  verifying the so-called entanglement relation:

$$(x \succ y) \prec z = x \succ (y \prec z), \quad \forall x, y, z \in L \quad (1)$$

Thus, an  $L$ -algebra is an  $\Omega$ -algebra where  $\Omega = \{\prec, \succ\}$ .

In the following, we always assume that  $\Omega = \{\prec, \succ\}$ .

Let  $k(X, \Omega)$  be the free  $\Omega$ -algebra generated by  $X$ . Let

$$S = \{(x \succ y) \prec z - x \succ (y \prec z) \mid x, y, z \in (X, \Omega)\}.$$

Then  $L(X) = k(X, \Omega|S) = k(X, \Omega)/Id(S)$  is clearly a free  $L$ -algebra generated by  $X$ .

We will order the set  $(X, \Omega)$ .

Let  $X$  be a well-ordered set. Denote  $|u|_X$  by  $|u|$ ,  $\succ$  by  $\delta_1$ , and  $\prec$  by  $\delta_2$ . Let  $\delta_1 < \delta_2$ . For any  $u \in (X, \Omega)$ , if  $u = x \in X$ , denote by

$$wt(u) = (1, x);$$

if  $u = \delta_i(u_1, u_2)$  for some  $u_1, u_2 \in (X, \Omega)$ , denote by

$$wt(u) = (|u|, \delta_i, u_1, u_2).$$

For any  $u, v \in (X, \Omega)$ , define

$$u > v \iff wt(u) > wt(v) \quad \text{lexicographically} \quad (2)$$

by induction on  $|u| + |v|$ .

It is clear that  $>$  is a monomial ordering on  $(X, \Omega)$ .

We will use the ordering (2) on  $(X, \Omega)$  in sequel.

**Theorem 3.2** With the ordering (2) on  $(X, \Omega)$ ,

$$S = \{(x \succ y) \prec z - x \succ (y \prec z) \mid x, y, z \in (X, \Omega)\}$$

is a Gröbner-Shirshov basis in  $k(X, \Omega)$ .

**Proof.** All the possible ambiguities of compositions of  $\Omega$ -polynomials in  $S$  are:

$$i) (x|_{(a \succ b) \prec c} \succ y) \prec z \quad ii) (x \succ y|_{(a \succ b) \prec c}) \prec z \quad iii) (x \succ y) \prec z|_{(a \succ b) \prec c}$$

where  $a, b, c, x, y, z \in (X, \Omega)$ . It is easy to check that all these compositions are trivial. Here, for example, we just check  $i)$ . Others are similarly proved. Let

$$f(x, y, z) = (x \succ y) \prec z - x \succ (y \prec z).$$

Then

$$\begin{aligned} & (f(x|_{(a \succ b) \prec c}, y, z), f(a, b, c))_{(x|_{(a \succ b) \prec c} \succ y) \prec z} \\ &= -x|_{(a \succ b) \prec c} \succ (y \prec z) + (x|_{a \succ (b \prec c)} \succ y) \prec z \\ &\equiv -x|_{a \succ (b \prec c)} \succ (y \prec z) + x|_{a \succ (b \prec c)} \succ (y \prec z) \\ &\equiv 0 \mod(S, (x|_{(a \succ b) \prec c} \succ y) \prec z). \end{aligned}$$

□

An  $\Omega$ -word  $u$  is a normal word if  $u$  is one of the following:

- i)  $u = x$ , where  $x \in X$ .
- ii)  $u = v \succ w$ , where  $v$  and  $w$  are normal words.
- iii)  $u = v \prec w$  with  $v \neq v_1 \succ v_2$ , where  $v_1, v_2, v, w$  are normal words.

We denote  $u$  by  $[u]$  if  $u$  is a normal word.

Now, by Theorem 2.4, we have the following corollary.

**Corollary 3.3** *The set*

$$\text{Irr}(S) = \{u \in (X, \Omega) \mid u \neq v|_{(a \succ b) \prec c}, \ a, b, c \in (X, \Omega), \ v \text{ is a } \star\text{-}\Omega\text{-word}\}$$

*is a  $k$ -basis of the free  $L$ -algebra  $L(X) = k(X, \Omega|S)$ . Moreover,  $\text{Irr}(S)$  consists of all normal words in  $(X, \Omega)$ .*

The following proposition follows from (1) and Corollary 3.3.

**Proposition 3.4** *For any  $\Omega$ -word  $u$ , there exists a unique normal word  $[u]$  such that  $u = [u]$  in  $L(X)$ .*

We denote the set of all the normal words by  $N$ , i.e.,  $N = \text{Irr}(S)$ . Then, the free  $L$ -algebra has an expression  $L(X) = kN = \{\sum \alpha_i u_i \mid \alpha_i \in k, u_i \in N\}$  with a  $k$ -basis  $N$  and the operations  $\prec, \succ$ : for any  $u, v \in N$ ,

$$u \prec v = [u \prec v], \quad u \succ v = [u \succ v].$$

Clearly,  $[u \succ v] = u \succ v$  and

$$[u \prec v] = \begin{cases} u \prec v & \text{if } u = u_1 \prec u_2, \text{ or } u \in X, \\ u_1 \succ [u_2 \prec v] & \text{if } u = u_1 \succ u_2. \end{cases}$$

## 4 Composition-Diamond lemma for $L$ -algebras

In this section, we establish Composition-Diamond lemma for  $L$ -algebras. Remind that  $\Omega = \{\prec, \succ\}$ .

We use still the ordering (2) on  $N$  defined as before.

Let  $u$  be a  $\star$ - $\Omega$ -word and  $s \in L(X)$ . Then we call  $u|_s = u|_{\star \mapsto s}$  an  $s$ -word in  $L(X)$ . Let  $u$  be a  $(\star_1, \star_2)$ - $\Omega$ -word and  $s_1, s_2 \in L(X)$ . We call

$$u|_{s_1, s_2} = u|_{\star_1 \mapsto s_1, \star_2 \mapsto s_2}$$

an  $s_1$ - $s_2$ -word.

An  $s$ -word  $u|_s$  is called a normal  $s$ -word if  $u|_s \in N$ .

It is noted that the  $s$ -word  $u|_s$  is a normal  $s$ -word if and only if  $\overline{u|_s} = u|_{\overline{s}}$  as  $\Omega$ -words.

We will prove that the ordering (2) on  $N$  is monomial in the sense that for any  $\star$ - $\Omega$ -word  $w$  and any  $u, v \in N$ ,  $u > v$  implies  $[w|_u] > [w|_v]$ .

**Lemma 4.1** *The ordering (2) on  $N$  is monomial.*

**Proof.** To prove this lemma, we only need to prove that, for any  $u, v, w \in N$ ,  $u > v$  implies  $[u \succ w] > [v \succ w]$ ,  $[w \succ u] > [w \succ v]$ ,  $[w \prec u] > [w \prec v]$  and  $[u \prec w] > [v \prec w]$ . But only the final case needs to check, since the other three cases are just obvious.

It is clear that  $u$  has a unique expression:

$$u = u_1 \succ (u_2 \succ (\cdots \succ (u_{n-1} \succ u_n) \cdots))$$

where  $n \geq 1$  and  $u_n \neq a \succ b$  for any  $a, b \in N$ . For example, if  $u = a \prec b$ , then  $n = 1$ . Let

$$v = v_1 \succ (v_2 \succ (\cdots \succ (v_{m-1} \succ v_m) \cdots))$$

where  $m \geq 1$  and  $v_m \neq a \succ b$  for any  $a, b \in N$ . Then

$$[u \prec w] = u_1 \succ (u_2 \succ (\cdots \succ (u_{n-1} \succ (u_n \prec w)) \cdots)),$$

$$[v \prec w] = v_1 \succ (v_2 \succ (\cdots \succ (v_{m-1} \succ (v_m \prec w)) \cdots)).$$

If  $|u| > |v|$ ,  $[u \prec w] > [v \prec w]$  is obvious. We assume that  $|u| = |v|$ . Since  $u > v$ , there must be a  $j$  such that  $u_j > v_j$ , and for any  $i < j$ ,  $u_i = v_i$ . So

$$(u_j \succ (\cdots \succ (u_{n-1} \succ (u_n \prec w)) \cdots)) > (v_j \succ (\cdots \succ (v_{m-1} \succ (v_m \prec w)) \cdots)).$$

It follows that  $[u \prec w] > [v \prec w]$ . □

Now we define compositions of polynomials in  $L(X)$ .

**Definition 4.2** *Let  $f, g \in L(X)$  with  $f$  and  $g$  monic.*

1) *Composition of right multiplication.*

*If  $\bar{f} = u_1 \succ u_2$  for some  $u_1, u_2 \in N$ , then for any  $v \in N$ ,  $f \prec v$  is called the composition of right multiplication.*

2) *Composition of inclusion.*

*If  $w = \bar{f} = u|_{\bar{g}}$  where  $u|_{\bar{g}}$  is a normal  $g$ -word, then*

$$(f, g)_w = f - u|_g$$

*is called a composition of inclusion.*

**Definition 4.3** *Let  $S \subset L(X)$  be a monic set and  $f, g \in S$ .*

*The composition of right multiplication  $f \prec v$  is called trivial modulo  $S$ , denoted by  $f \prec v \equiv 0 \pmod{S}$ , if*

$$f \prec v = \sum \alpha_i u_i|_{s_i},$$

*where each  $\alpha_i \in k$ ,  $s_i \in S$ ,  $u_i|_{s_i}$  normal  $s_i$ -word, and  $u_i|_{\bar{s}_i} \leq \overline{f \prec v}$ .*

*The composition of inclusion  $(f, g)_w$  is called trivial modulo  $(S, w)$ , denoted by  $(f, g)_w \equiv 0 \pmod{(S, w)}$ , if*

$$(f, g)_w = \sum \alpha_i u_i|_{s_i},$$

*where each  $\alpha_i \in k$ ,  $s_i \in S$ ,  $u_i|_{s_i}$  normal  $s_i$ -word, and  $u_i|_{\bar{s}_i} < w$ .*

*$S$  is called a Gröbner-Shirshov basis in  $L(X)$  if any composition of polynomials in  $S$  is trivial.*



**Lemma 4.4** *Let  $S \subset L(X)$  and  $u|_s$  an  $s$ -word,  $s \in S$ . Assume that each composition of right multiplication in  $S$  is trivial modulo  $S$ . Then,  $u|_s$  has a presentation:*

$$u|_s = \sum \alpha_i u_i|_{s_i},$$

where each  $\alpha_i \in k$ ,  $s_i \in S$ ,  $u_i|_{s_i}$  normal  $s_i$ -word and  $u_i|_{\overline{s_i}} \leq \overline{u|_s}$ .

**Proof.** By Proposition 3.4, we may assume that  $u = [u] \in N$ . We prove the result by induction on  $|u|$ .

If  $|u| = 1$ ,  $u|_s = s$  is a normal  $s$ -word.

If  $|u| > 1$ , we have four cases to consider:

1)  $u|_s = w \succ v|_s$ ,

2)  $u|_s = v|_s \succ w$ ,

3)  $u|_s = w \prec v|_s$ ,

4)  $u|_s = v|_s \prec w$ .

Since  $w$  is a subword of the normal word  $u$ ,  $w$  is also an normal word. Since  $|v| < |u|$ ,  $v|_s = \sum \alpha_j u_j|_{s_j}$ , where each  $\alpha_j \in k$ ,  $s_j \in S$ ,  $u_j|_{s_j}$  normal  $s_j$ -word and  $u_j|_{\overline{s_j}} \leq \overline{v|_s}$ . Moreover, we may assume that  $v|_s$  is a normal  $s$ -word.

In Case 1), 2) and 3),  $\Omega$ -word  $u|_{\overline{s}} \in N$  and there is nothing to prove. We only consider Case 4).

In Case 4), if  $|v| = 1$  and  $\overline{s} = a \prec b$  for some  $a, b \in N$ , then  $\Omega$ -word  $u|_{\overline{s}} \in N$ . If  $|v| = 1$  and  $\overline{s} = a \succ b$ , then by right multiplication,  $u|_s = s \prec w = \sum \alpha_i u_i|_{s_i}$ , where each  $\alpha_i \in k$ ,  $s_i \in S$ ,  $u_i|_{s_i}$  normal  $s_i$ -word and  $u_i|_{\overline{s_i}} \leq \overline{u|_s}$ .

For  $|v| > 1$ , since  $u = [u] \in N$ ,  $v = a \prec b$  for some  $a, b \in N$ . Then by  $v|_{\overline{s}} \in N$  we have  $u|_{\overline{s}} = v|_{\overline{s}} \prec w \in N$  as  $\Omega$ -words. So,  $u|_s$  is a normal  $s$ -word.

The lemma is proved.  $\square$

**Lemma 4.5** *Let  $S$  be a Gröbner-Shirshov basis in  $L(X)$ ,  $s_1, s_2 \in S$ ,  $u_1|_{s_1}$  and  $u_2|_{s_2}$  be normal  $s_1$ -word and normal  $s_2$ -word respectively such that  $w = u_1|_{\overline{s_1}} = u_2|_{\overline{s_2}}$ . Then,*

$$u_1|_{s_1} \equiv u_2|_{s_2} \mod(S, w),$$

where  $u_1|_{s_1} \equiv u_2|_{s_2} \mod(S, w)$  means  $u_1|_{s_1} - u_2|_{s_2} = \sum \alpha_i u_i|_{s_i}$  for some  $\alpha_i \in k$ ,  $s_i \in S$ ,  $u_i|_{s_i}$  normal  $s_i$ -word such that  $u_i|_{\overline{s_i}} < w$ .

**Proof.** Since the operations  $\prec$  and  $\succ$  are not associative, there are only two cases to consider.

1)  $\overline{s_1}$  and  $\overline{s_2}$  are disjoint in  $w$ . Then there exists a  $(\star_1, \star_2)$ - $\Omega$ -word  $\Pi$  such that

$$\Pi|_{\overline{s_1}, \overline{s_2}} = u_1|_{\overline{s_1}} = u_2|_{\overline{s_2}},$$

where  $\Pi|_{s_1, s_2}$  is an  $s_1$ - $s_2$ -word. Then

$$u_1|_{s_1} - u_2|_{s_2} = \Pi|_{s_1, \overline{s_2}} - \Pi|_{\overline{s_1}, s_2} = -\Pi|_{s_1, s_2 - \overline{s_2}} + \Pi|_{s_1 - \overline{s_1}, s_2}.$$

Let

$$\begin{aligned}\Pi|_{s_1-\overline{s_1},s_2} &= \sum_r \alpha_{2r} u_{2r}|_{s_2}, \\ -\Pi|_{s_1,s_2-\overline{s_2}} &= \sum_t \alpha_{1t} u_{1t}|_{s_1}.\end{aligned}$$

Since  $S$  is a Gröbner-Shirshov basis, by Lemma 4.4, we have

$$u_{2r}|_{s_2} = \sum_n \beta_{rn} v_{rn}|_{s_{rn}} \quad \text{and} \quad u_{1t}|_{s_1} = \sum_m \beta_{tm} v_{tm}|_{s_{tm}},$$

where  $\beta_{rn}, \beta_{tm} \in k$ ,  $s_{rn}, s_{tm} \in S$ ,  $v_{rn}|_{s_{rn}}$  normal  $s_{rn}$ -word,  $v_{tm}|_{s_{tm}}$  normal  $s_{tm}$ -word, and  $v_{tm}|_{\overline{s_{tm}}} \leq \overline{u_{1t}|_{s_1}}$ ,  $v_{rn}|_{\overline{s_{rn}}} \leq \overline{u_{2r}|_{s_2}}$ . Then

$$u_1|_{s_1} - u_2|_{s_2} = \sum_{t,m} \alpha_{1t} \beta_{tm} v_{tm}|_{s_{tm}} + \sum_{r,n} \alpha_{2r} \beta_{rn} v_{rn}|_{s_{rn}}.$$

Since

$$\begin{aligned}v_{tm}|_{\overline{s_{tm}}} &\leq \overline{u_{1t}|_{s_1}} \leq \overline{\Pi|_{s_1,s_2-\overline{s_2}}} = \Pi|_{\overline{s_1},s_2-\overline{s_2}} < \Pi|_{\overline{s_1},\overline{s_2}} = w \quad \text{and} \\ v_{rn}|_{\overline{s_{rn}}} &\leq \overline{u_{2r}|_{s_2}} \leq \overline{\Pi|_{s_1-\overline{s_1},s_2}} = \Pi|_{\overline{s_1-\overline{s_1}},\overline{s_2}} < \Pi|_{\overline{s_1},\overline{s_2}} = w,\end{aligned}$$

we have

$$u_1|_{s_1} \equiv u_2|_{s_2} \pmod{(S, w)}.$$

2) One of  $\overline{s_1}, \overline{s_2}$  is contained in the other. We may assume that  $\overline{s_2}$  is contained in  $\overline{s_1}$ . Then  $\overline{s_1} = u|_{\overline{s_2}}$  for some normal  $s_2$ -word  $u|_{s_2}$ . So

$$w = u_1|_{\overline{s_1}} = u_1|_{u|_{\overline{s_2}}}$$

and

$$u_1|_{s_1} - u_2|_{s_2} = u_1|_{s_1} - u_1|_{u|_{s_2}} = u_1|_{s_1-u|_{s_2}}.$$

Since  $S$  is a Gröbner-Shirshov basis in  $L(X)$ , we have

$$s_1 - u|_{s_2} = \sum_t \alpha_t v_t|_{s_t}$$

where each  $\alpha_t \in k$ ,  $s_t \in S$ ,  $v_t|_{s_t}$  normal  $s_t$ -word, and  $v_t|_{\overline{s_t}} < \overline{s_1}$ . Let  $u_1|_{v_t|_{s_t}} = u_{1t}|_{s_t}$ . Then by Lemma 4.4, we have

$$u_{1t}|_{s_t} = \sum_n \beta_{tn} v_{tn}|_{s_{tn}}$$

where each  $\beta_n \in k$ ,  $s_{tn} \in S$ ,  $v_{tn}|_{s_{tn}}$  normal  $s_{tn}$ -word, and  $v_{tn}|_{\overline{s_{tn}}} \leq \overline{u_{1t}|_{s_t}}$ . So

$$u_1|_{s_1} - u_2|_{s_2} = \sum_t \alpha_t u_1|_{v_t|_{s_t}} = \sum_t \alpha_t u_{1t}|_{s_t} = \sum_{t,n} \alpha_t \beta_{tn} v_{tn}|_{s_{tn}}$$

with

$$v_{tn}|_{\overline{s_{tn}}} \leq \overline{u_{1t}|_{s_t}} < u_1|_{\overline{s_1}} = w.$$

It follows that

$$u_1|_{s_1} \equiv u_2|_{s_2} \pmod{(S, w)}.$$

The proof is completed.  $\square$

**Theorem 4.6** (*Composition-Diamond lemma for  $L$ -algebras*) Let  $S \subset L(X)$  be a monic set and the ordering  $>$  defined on  $N$  as (2). Then the following statements are equivalent:

- I)  $S$  is a Gröbner-Shirshov basis in  $L(X)$ .
- II)  $f \in Id(S) \Rightarrow \bar{f} = u|_{\bar{s}}$  for some normal  $s$ -word  $u|_s$ ,  $s \in S$ .
- III) The set

$$Irr(S) = \{u \in N \mid u \neq v|_{\bar{s}}, s \in S, v|_s \text{ is a normal } s\text{-word}\}$$

is a linear basis of the  $L$ -algebra  $L(X|S) = L(X)/Id(S)$ , where  $Id(S)$  is the ideal of  $L(X)$  generated by  $S$ .

**Proof.**  $I) \Rightarrow II)$ . Let  $S$  be a Gröbner-Shirshov basis and  $0 \neq f \in Id(S)$ . By Lemma 4.4, we can assume that

$$f = \sum_{i=1}^n \alpha_i u_i|_{s_i},$$

where each  $\alpha_i \in k$ ,  $s_i \in S$  and  $u_i|_{s_i}$  normal  $s_i$ -word. Let

$$w_i = u_i|_{\bar{s}_i}, w_1 = w_2 = \dots = w_l > w_{l+1} \geq \dots$$

We prove the theorem by induction on  $l$  and  $w_1$ .

If  $l = 1$ , then  $\bar{f} = \overline{u_1|_{s_1}} = u_1|_{\bar{s}_1}$  and the result holds.

Assume that  $l \geq 2$ . Then

$$\alpha_1 u_1|_{s_1} + \alpha_2 u_2|_{s_2} = (\alpha_1 + \alpha_2) u_1|_{s_1} + \alpha_2 (u_2|_{s_2} - u_1|_{s_1}).$$

By Lemma 4.5, we have

$$u_2|_{s_2} \equiv u_1|_{s_1} \pmod{(S, w_1)}.$$

Now, the remainder proof of the theorem is almost the same as one in Theorem 2.4.  $\square$

## 5 Applications

In this section, we use the Theorem 4.6 to prove four embedding theorems for  $L$ -algebras. We give Gröbner-Shirshov bases of a free dialgebra and the free product of two  $L$ -algebras, respectively and then the norm forms are obtained for such algebras.

Let  $X$  be a set. Then we denote the set of all the normal words in the free  $L$ -algebra  $L(X)$  defined as before by  $N(X)$ .

Denote by  $\mathbb{N}^+$  the set of all positive natural numbers and  $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$ .

The following lemma is straightforward.

**Lemma 5.1** Let  $A$  be an  $L$ -algebra over a field  $k$  with a  $k$ -basis  $X = \{x_i \mid i \in I\}$ . Then  $A$  has a representation  $A = L(X|S)$ , where  $S = \{x_i \prec x_j - \{x_i \prec x_j\}, x_i \succ x_j - \{x_i \succ x_j\} \mid i, j \in I\}$ ,  $\{x_i \succ x_j\}$  and  $\{x_i \prec x_j\}$  are linear combinations of  $x_t \in X$ . Moreover, with the ordering (2) on  $N(X)$ ,  $S$  is a Gröbner-Shirshov basis in  $L(X)$ .

## 5.1 Gröbner-Shirshov basis for the free product of two $L$ -algebras

**Definition 5.2** Let  $L_1, L_2$  be  $L$ -algebras. Then an  $L$ -algebra  $L_1 * L_2$  with two  $L$ -algebra homomorphisms  $\varepsilon_i : L_i \rightarrow L_1 * L_2$ ,  $i = 1, 2$  is called the free product of  $L_1$  and  $L_2$ , if the following diagram commutes:

$$\begin{array}{ccccc} L_1 & \xrightarrow{\varepsilon_1} & L_1 * L_2 & \xleftarrow{\varepsilon_2} & L_2 \\ & \searrow \forall f_1 & \downarrow \exists! f & \swarrow \forall f_2 & \\ & & L & & \end{array}$$

where  $L$  is any  $L$ -algebra and  $f_1, f_2$  are  $L$ -algebra homomorphisms. It means that  $(L_1 * L_2, (\varepsilon_1, \varepsilon_2))$  is a universal arrow in the sense of S. MacLane [9].

For  $L$ -algebras  $L_1$  and  $L_2$ , let  $X = \{x_i | i \in I\}$  and  $Y = \{y_j | j \in J\}$  are  $k$ -bases of  $L_1$  and  $L_2$ , respectively. Then  $L_1 = L(X|S_1)$  and  $L_2 = L(Y|S_2)$ , where

$$\begin{aligned} S_1 &= \{x_i \prec x_j - \{x_i \prec x_j\}, x_i \succ x_j - \{x_i \succ x_j\} | i, j \in I\}, \\ S_2 &= \{y_i \prec y_j - \{y_i \prec y_j\}, y_i \succ y_j - \{y_i \succ y_j\} | i, j \in J\}. \end{aligned}$$

It is clear that  $L_1 * L_2 = L(X_1 \cup X_2 | S_1 \cup S_2)$ .

Let  $X \cup Y$  be a well-ordered set. Let  $S = S_1 \cup S_2 \cup F_1 \cup F_2$ , where

$$\begin{aligned} F_1 &= \{f_{1(ij)} = x_i \succ (((x_j \prec u) \prec v_1) \prec \dots) \prec v_n - \\ &\quad (((\{x_i \succ x_j\} \prec u) \prec v_1) \prec \dots) \prec v_n \mid n \in \mathbb{N}, u, v_l \in N(X \cup Y), \\ &\quad u \in \text{Irr}(S_1 \cup S_2) - X, v_l \in \text{Irr}(S_1 \cup S_2), l = 1, \dots, n\}, \\ F_2 &= \{f_{2(ij)} = y_i \succ (((y_j \prec u) \prec v_1) \prec \dots) \prec v_n - \\ &\quad (((\{y_i \succ y_j\} \prec u) \prec v_1) \prec \dots) \prec v_n \mid n \in \mathbb{N}, u, v_l \in N(X \cup Y), \\ &\quad u \in \text{Irr}(S_1 \cup S_2) - Y, v_l \in \text{Irr}(S_1 \cup S_2), l = 1, \dots, n\}. \end{aligned}$$

Then we have the following theorem.

**Theorem 5.3** With the ordering (2) on  $N(X \cup Y)$ ,  $S$  is a Gröbner-Shirshov basis of  $L_1 * L_2 = L(X \cup Y | S_1 \cup S_2)$ .

**Proof.** All the possible ambiguities  $w$  of composition of inclusion in  $S$  is  $\overline{f_{1(ij)}}|_{\overline{f'_{1(i'j')}}}$ ,  $\overline{f_{2(ij)}}|_{\overline{f'_{2(i'j')}}}$ ,  $\overline{f_{1(ij)}}|_{\overline{f'_{2(i'j')}}}$  and  $\overline{f_{2(ij)}}|_{\overline{f'_{1(i'j')}}}$ , where  $\overline{f'_{1(i'j')}}$  ( $\overline{f'_{1(i'j')}}$ ,  $\overline{f'_{2(i'j')}}$ ,  $\overline{f'_{2(i'j')}}$  respectively) is a subword of  $u$  or some  $v_r$  in  $\overline{f_{1(ij)}}$  ( $\overline{f_{2(ij)}}$ ,  $\overline{f_{1(ij)}}$ ,  $\overline{f_{2(ij)}}$  respectively).

All the possible compositions of right multiplication are:  $(x_i \succ x_j - \{x_i \succ x_j\}) \prec u$ ,  $(y_i \succ y_j - \{y_i \succ y_j\}) \prec u$ ,  $f_{1(ij)} \prec u$  and  $f_{2(ij)} \prec u$  where  $u \in N(X \cup Y)$ .

All the compositions are trivial by a similar analysis in Theorem 5.8.  $\square$

For any  $u \in N(X \cup Y)$ , we have  $u \in Irr(S)$  if and only if  $u$  is one of the following:

- I)  $u \in X \cup Y$ ,
- II)  $u \in (X \prec Y) \cup (X \succ Y) \cup (Y \prec X) \cup (Y \succ X)$ , where, for example,  
 $X \prec Y = \{x \prec y | x \in X, y \in Y\}$ ,
- III) for  $|u| > 2$ , there are four cases:
  - 1)  $u = x_i \succ w$ , with  $w \neq ((x_j \prec w_1) \prec \dots) \prec w_n$ , where  $x_i, x_j \in X$ ,  $w, w_t \in Irr(S)$ ,
  - 2)  $u = y_i \succ w$ , with  $w \neq ((y_j \prec w_1) \prec \dots) \prec w_n$ , where  $y_i, y_j \in Y$ ,  $w, w_t \in Irr(S)$ ,
  - 3)  $u = v \succ w$ , with  $|v| \geq 2$ , where  $v, w \in Irr(S)$ ,
  - 4)  $u = v \prec w$ , with  $v \neq v_1 \succ v_2$ , where  $v, w, v_1, v_2 \in Irr(S)$ .

**Corollary 5.4** *By Theorem 4.6,  $Irr(S)$  is a  $k$ -basis of  $L_1 * L_2$ .*

## 5.2 Gröbner-Shirshov basis for a free dialgebra

**Definition 5.5** *Let  $k$  be a field. A  $k$ -linear space  $D$  equipped with two bilinear multiplications  $\succ$  and  $\prec$  is called a dialgebra, if both  $\succ$  and  $\prec$  are associative and*

$$\begin{aligned} a \prec (b \succ c) &= a \prec b \prec c \\ (a \prec b) \succ c &= a \succ b \succ c \\ a \succ (b \prec c) &= (a \succ b) \prec c \end{aligned}$$

for any  $a, b, c \in D$ .

Let  $D(X)$  be the free dialgebra generated by  $X$ . Then it is clear that  $D(X)$  is also an  $L$ -algebra and  $D(X) = L(X|S)$ , where  $S$  consists of

$$\begin{aligned} F_1 &= \{a \prec (b \prec c) - a \prec (b \succ c) \mid a, b, c \in N(X)\}, \\ F_2 &= \{(a \prec b) \succ c - a \succ (b \succ c) \mid a, b, c \in N(X)\}, \\ F_3 &= \{(a \prec b) \prec c - a \prec (b \succ c) \mid a, b, c \in N(X)\}, \\ F_4 &= \{(a \succ b) \succ c - a \succ (b \succ c) \mid a, b, c \in N(X)\}. \end{aligned}$$

Denote by

$$F_5 = \{f_{5(n)} = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (a_{n+2} \prec a_{n+3}) \dots)) - a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (a_{n+2} \succ a_{n+3}) \dots)) \mid n \in \mathbb{N}^+, a_i \in N(X), i = 1, 2, \dots, n+3\}.$$

Equipping with the above concepts, we have the following theorem.

**Theorem 5.6** *Let  $X$  be a well-ordered set. With the ordering (2) on  $N(X)$ ,  $S_1 = S \cup F_5$  is a Gröbner-Shirshov basis in  $L(X)$ .*

**Proof.** The possible compositions of right multiplication are:  $f \prec u$ , where  $f \in F_2 \cup F_4$ ,  $u \in N(X)$ . If  $f \in F_2$ , we have

$$\begin{aligned}
f \prec u &= ((a \prec b) \succ c) \prec u - (a \succ (b \succ c)) \prec u \\
&\equiv (a \prec b) \succ (c \prec u) - a \succ ((b \succ c) \prec u) \\
&\equiv a \succ (b \succ (c \prec u)) - a \succ (b \succ (c \prec u)) \\
&\equiv 0 \mod(S).
\end{aligned}$$

Similarly,  $f \prec u \equiv 0 \mod(S)$  where  $f \in F_4$ .

For compositions of inclusion, if some  $\overline{f_i}$  is a subword of  $a$  ( $b$ ,  $c$ ,  $a_i$  respectively) in some  $\overline{f_j}$ , the composition  $(f_j, f_i)_{\overline{f_j}}$  is trivial by a similar proof in Theorem 3.2.

All the other ambiguities  $w$  of compositions of inclusion in  $S_1$  are:

- 1)  $w = a \prec (b \prec (c \prec d)) = a \prec \overline{f_1} = \overline{f'_1}$ , where  $f_1, f'_1 \in F_1$ ,
- 2)  $w = (a \prec (b \prec c)) \succ d = \overline{f_1} \succ d = \overline{f_2}$ , where  $f_1 \in F_1$ ,  $f_2 \in F_2$ ,
- 3)  $w = a \prec ((b \prec c) \prec d) = a \prec \overline{f_3} = \overline{f_1}$ , where  $f_1 \in F_1$ ,  $f_3 \in F_3$ ,
- 4)  $w = (a \prec (b \prec c)) \prec d = \overline{f_1} \prec d = \overline{f_3}$ , where  $f_1 \in F_1$ ,  $f_3 \in F_3$ ,
- 5)  $w = ((a \prec b) \prec c) \succ d = \overline{f_3} \succ d = \overline{f_2}$ , where  $f_2 \in F_2$ ,  $f_3 \in F_3$ ,
- 6)  $w = ((a \prec b) \succ c) \succ d = \overline{f_2} \succ d = \overline{f_4}$ , where  $f_2 \in F_2$ ,  $f_4 \in F_4$ ,
- 7)  $w = ((a \prec b) \prec c) \prec d = \overline{f_3} \prec d = \overline{f'_3}$ , where  $f_3, f'_3 \in F_3$ ,
- 8)  $w = ((a \succ b) \succ c) \succ d = \overline{f_4} \succ d = \overline{f'_4}$ , where  $f_4, f'_4 \in F_4$ ,
- 9)  $w = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (a_{i+2} \prec (a_{i+3} \prec a_{i+4})) \dots)) = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (\overline{f_1}) \dots)) = \overline{f_{5(i)}}$ , where  $f_1 \in F_1$ ,  $f_{5(i)} \in F_5$ ,
- 10)  $w = a \prec (a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (a_{i+2} \prec a_{i+3}) \dots))) = a \prec \overline{f_{5(i)}} = \overline{f_1}$ , where  $f_1 \in F_1$ ,  $f_{5(i)} \in F_5$ ,
- 11)  $w = (a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (a_{i+2} \prec a_{i+3}) \dots))) \succ c = \overline{f_{5(i)}} \succ c = \overline{f_2}$ , where  $f_2 \in F_2$ ,  $f_{5(i)} \in F_5$ ,
- 12)  $w = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ ((a \prec b) \succ (a_{i+2} \prec a_{i+3}) \dots))) = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (\overline{f_2} \prec a_{i+3}) \dots)) = \overline{f_{5(i)}}$ , where  $f_2 \in F_2$ ,  $f_{5(i)} \in F_5$ ,
- 13)  $w = (a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (a_{i+2} \prec a_{i+3}) \dots))) \prec c = \overline{f_{5(i)}} \prec c = \overline{f_3}$ , where  $f_3 \in F_3$ ,  $f_{5(i)} \in F_5$ ,
- 14)  $w = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ ((a_{i+2} \prec a_{i+3}) \prec a_{i+4}) \dots)) = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (\overline{f_3}) \dots)) = \overline{f_{5(i)}}$ , where  $f_3 \in F_3$ ,  $f_{5(i)} \in F_5$ ,
- 15)  $w = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ ((a \succ b) \succ (a_{i+2} \prec a_{i+3}) \dots))) = a_1 \prec (a_2 \succ (a_3 \succ \dots \succ (\overline{f_4} \prec a_{i+3}) \dots)) = \overline{f_{5(i)}}$ , where  $f_4 \in F_4$ ,  $f_{5(i)} \in F_5$ ,
- 16)  $w = a_1 \prec (a_2 \succ \dots \succ (a_{i+1} \succ (\overline{b_1} \prec (b_2 \succ \dots \succ (b_{j+2} \prec b_{j+3}) \dots))) \dots) = a_1 \prec (a_2 \succ \dots \succ (a_{i+1} \succ \overline{f_{5(j)}}) \dots) = \overline{f_{5(i)}}$ , where  $f_{5(i)}, f_{5(j)} \in F_5$ .

These compositions are all trivial. Here, for example, we just check 1), 9) and 16). Others are similarly proved.

1) Let  $w = a \prec (b \prec (c \prec d)) = a \prec \overline{f_1} = \overline{f'_1}$ . We have

$$\begin{aligned} (f'_1, f_1)_w &\equiv a \prec (b \succ (c \prec d)) - a \prec (b \prec (c \succ d)) \\ &\equiv a \prec (b \succ (c \prec d)) - a \prec (b \succ (c \succ d)) \\ &\equiv f_{5(1)} \\ &\equiv 0 \mod(S_1, w). \end{aligned}$$

9) Let  $w = a_1 \prec (a_2 \succ (a_3 \succ \cdots \succ (a_{i+2} \prec (a_{i+3} \prec a_{i+4})) \cdots)) = a_1 \prec (a_2 \succ (a_3 \succ \cdots \succ (\overline{f_1}) \cdots)) = \overline{f_{5(i)}}$ . We have

$$\begin{aligned} (f_1, f_{5(i)})_w &\equiv a_1 \prec (a_2 \succ (a_3 \succ \cdots \succ (a_{i+2} \succ (a_{i+3} \prec a_{i+4})) \cdots)) \\ &\quad - a_1 \prec (a_2 \succ (a_3 \succ \cdots \succ (a_{i+2} \prec (a_{i+3} \succ a_{i+4})) \cdots)) \\ &\equiv a_1 \prec (a_2 \succ (a_3 \succ \cdots \succ (a_{i+2} \succ (a_{i+3} \prec a_{i+4})) \cdots)) \\ &\quad - a_1 \prec (a_2 \succ (a_3 \succ \cdots \succ (a_{i+2} \succ (a_{i+3} \succ a_{i+4})) \cdots)) \\ &\equiv f_{5(i+1)} \\ &\equiv 0 \mod(S_1, w). \end{aligned}$$

16) Let  $w = a_1 \prec (a_2 \succ \cdots \succ (a_{i+1} \succ (b_1 \prec (b_2 \succ \cdots \succ (b_{j+2} \prec b_{j+3}) \cdots)) \cdots)) = a_1 \prec (a_2 \succ \cdots \succ (a_{i+1} \succ \overline{f_{5(j)}}) \cdots) = \overline{f_{5(i)}}$ . We have

$$\begin{aligned} (f_{5(i)}, f_{5(j)})_w &\equiv a_1 \prec (a_2 \succ \cdots \succ (a_{i+1} \succ (b_1 \succ (b_2 \succ \cdots \succ (b_{j+2} \prec b_{j+3}) \cdots)) \cdots)) \\ &\quad - a_1 \prec (a_2 \succ \cdots \succ (a_{i+1} \succ (b_1 \prec (b_2 \succ \cdots \succ (b_{j+2} \succ b_{j+3}) \cdots)) \cdots)) \\ &\equiv 0 \mod(S_1, w). \end{aligned}$$

Then  $S_1$  is a Gröbner-Shirshov basis in  $L(X)$ . □

The following corollary follows from Theorems 4.6 and 5.6.

**Corollary 5.7** *The set  $\text{Irr}(S_1) = \{u = x_{-m} \succ (x_{-m+1} \succ \cdots \succ (x_0 \prec (x_1 \succ \cdots \succ (x_{n-1} \succ x_n) \cdots))) \mid n, m \in \mathbb{N}, x_i \in X\}$  is a  $k$ -basis of the free dialgebra  $D(X) = L(X|S)$ .*

**Remark:** In [8], Loday gives a  $k$ -basis of the free dialgebra  $D(X)$ , which is  $\{x_{-m} \succ \cdots \succ x_{-1} \succ x_0 \prec x_1 \prec \cdots \prec x_n \mid n, m \in \mathbb{N}, x_i \in X\}$ . It is easy to see that our  $k$ -basis is just the same as in [8] by using relation  $a \prec (b \prec c) = a \prec (b \succ c)$ .

### 5.3 Embedding theorems for $L$ -algebras

**Theorem 5.8** *Every countably generated  $L$ -algebra over a field  $k$  can be embedded into a two-generated  $L$ -algebra.*

**Proof.** Let  $A$  be a countably generated  $L$ -algebra. We may assume that  $A$  has a countable  $k$ -basis  $X = \{x_i \mid i \in \mathbb{N}^+\}$ . By Lemma 5.1,  $A = L(X|S)$ , where  $S = \{x_i \prec x_j - \{x_i \prec x_j\}, x_i \succ x_j - \{x_i \succ x_j\} \mid i, j \in \mathbb{N}^+\}$ .

Let  $Y = \{a, b\}$ ,

$$\begin{aligned}
F_1 &= \{f_{1(ij)} = x_i \succ x_j - \{x_i \succ x_j\} \mid i, j \in \mathbb{N}^+\}, \\
F_2 &= \{f_{2(ij)} = x_i \prec x_j - \{x_i \prec x_j\} \mid i, j \in \mathbb{N}^+\}, \\
F_3 &= \{f_{3(i)} = a \prec \underbrace{(b \prec (\cdots (b \prec b)))}_i - x_i \mid i, j \in \mathbb{N}^+\}, \\
F_4 &= \{f_{4(ij)} = x_i \succ (((x_j \prec u) \prec v_1) \prec \cdots) \prec v_n - \\
&\quad (((\{x_i \succ x_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid i, j \in \mathbb{N}^+, n \in \mathbb{N}, u, v_l \in N(X \cup Y), \\
&\quad u \in \text{Irr}(F_1 \cup F_2 \cup F_3) - X, v_l \in \text{Irr}(F_1 \cup F_2 \cup F_3), l = 1, \dots, n\}.
\end{aligned}$$

Let  $B = L(X \cup Y | S_1)$ , where  $S_1 = F_1 \cup F_2 \cup F_3 \cup F_4$ .

We want to prove that  $S_1$  is a Gröbner-Shirshov basis in  $L(X \cup Y)$  with ordering (2), where  $X \cup Y$  is a well-ordered set.

The only ambiguity  $w$  of composition of inclusion in  $S_1$  is  $\overline{f_{4(ij)}}|_{\overline{f'_{4(i'j')}}}$  where  $\overline{f'_{4(i'j')}}$  is a subword of  $u$  or some  $v_l$ . It is trivial by a similar proof in Theorem 3.2.

For right multiplication, only possible compositions of right multiplication are  $f_{m(ij)} \prec u$ ,  $m = 1, 4$ ,  $u \in N(X \cup Y)$ . We can use  $F_1$ ,  $F_2$  and  $F_3$  to reduce  $u$  into a  $v \in \text{Irr}(F_1 \cup F_2 \cup F_3)$  such that  $v \leq u$  and

$$f_{m(ij)} \prec u \equiv f_{m(ij)} \prec v \pmod{S_1}.$$

So  $f_{m(ij)} \prec u$  is trivial if  $f_{m(ij)} \prec v$  is trivial. Then we only need to consider right multiplication with  $u \in \text{Irr}(F_1 \cup F_2 \cup F_3)$ .

We consider the case of  $f_{1(ij)}$  first. For any  $u \in \text{Irr}(F_1 \cup F_2 \cup F_3)$ , if  $u \notin X$ , according to  $f_{4(ij)}$ ,

$$f_{1(ij)} \prec u = (x_i \succ x_j) \prec u - \{x_i \succ x_j\} \prec u \equiv 0 \pmod{S_1}.$$

If  $u = x_k \in X$ , then

$$\begin{aligned}
f_{1(ij)} \prec u &= (x_i \succ x_j) \prec u - \{x_i \succ x_j\} \prec u \\
&= x_i \succ (x_j \prec x_k) - \{x_i \succ x_j\} \prec x_k \\
&\equiv x_i \succ \{x_j \prec x_k\} - \{x_i \succ x_j\} \prec x_k \\
&\equiv \{x_i \succ \{x_j \prec x_k\}\} - \{\{x_i \succ x_j\} \prec x_k\} \\
&\equiv 0 \pmod{S_1}.
\end{aligned}$$

Now we consider the case of  $f_{4(ij)}$ . For any  $w \in \text{Irr}(F_1 \cup F_2 \cup F_3)$ ,

$$\begin{aligned}
f_{4(ij)} \prec w &= (x_i \succ (((x_j \prec u) \prec \cdots) \prec v_n)) \prec w - (((\{x_i \succ x_j\} \prec u) \prec \cdots) \prec v_n) \prec w \\
&= x_i \succ (((x_j \prec u) \prec \cdots) \prec v_n) \prec w - (((\{x_i \succ x_j\} \prec u) \prec \cdots) \prec v_n) \prec w \\
&\equiv 0 \pmod{S_1}.
\end{aligned}$$

So  $S_1$  is a Gröbner-Shirshov basis in  $L(X \cup Y)$ . By Theorem 4.6,  $A$  can be embedded into  $B$  which is generated by  $Y = \{a, b\}$ .  $\square$

**Theorem 5.9** *Every  $L$ -algebra over a field  $k$  can be embedded into a simple  $L$ -algebra.*



**Proof.** Let  $A$  be an  $L$ -algebra over  $k$  with a  $k$ -basis  $X = \{x_i | i \in I\}$ . By Lemma 5.1,  $A = L(X|S)$ , where  $S = \{x_i \prec x_j - \{x_i \prec x_j\}, x_i \succ x_j - \{x_i \succ x_j\} | i, j \in I\}$ . Let  $I$  be a well-ordered set. Then with the ordering (2),  $S$  is clearly a Gröbner-Shirshov basis in  $L(X)$ .

We well order the set of monic elements of  $A$ . Denote by  $T$  the set of indices for the resulting well-ordered set. Consider the set  $T^2 = \{(\theta, \sigma)\}$  and assign  $(\theta, \sigma) < (\theta', \sigma')$  if either  $\theta < \theta'$  or  $\theta = \theta'$  and  $\sigma < \sigma'$ . Then  $T^2$  is also a well-ordered set.

For each ordered pair of elements  $s_\theta, s_\sigma \in A$ ,  $\theta, \sigma \in T$ , introduce the letters  $x_{\theta\sigma}, y_{\theta\sigma}$ .

Let  $A_1$  be the  $L$ -algebra generated by

$$X_1 = \{x_i, y_{\theta\sigma}, x_{\varrho\tau} | i \in I, \theta, \sigma, \varrho, \tau \in T\}$$

and define the relation set  $S_1 = F_1 \cup F_2 \cup F_3 \cup F_4$ , where

$$\begin{aligned} F_1 &= \{x_i \succ x_j - \{x_i \succ x_j\} | i, j \in I\}, \\ F_2 &= \{x_i \prec x_j - \{x_i \prec x_j\} | i, j \in I\}, \\ F_3 &= \{x_{\theta\sigma} \prec (s_\theta \succ y_{\theta\sigma}) - s_\sigma | (\theta, \sigma) \in T^2\}, \\ F_4 &= \{x_i \succ (((x_j \prec u) \prec v_1) \prec \dots) \prec v_n) - (((\{x_i \succ x_j\} \prec u) \prec v_1) \prec \dots) \prec v_n \mid \\ &\quad i, j \in I, n \in \mathbb{N}, u, v_l \in L(X_1), u \in \text{Irr}(F_1 \cup F_2 \cup F_3) - X, \\ &\quad v_l \in \text{Irr}(F_1 \cup F_2 \cup F_3), l = 1, \dots, n\}. \end{aligned}$$

Then, by a similar proof of Theorem 5.8,  $S_1$  is a Gröbner-Shirshov basis in  $L(X_1)$  with the ordering (2), where  $x_i < y_{\theta\sigma} < x_{\varrho\tau}$ .

Thus by Theorem 4.6,  $A$  can be embedded into  $A_1$ . In  $A_1$  every monic element  $f_\theta$  of the subalgebra  $A$  generates an ideal containing algebra  $A$ .

By the same construction of the  $L$ -algebra  $A_1$  from  $A$ , we get the  $L$ -algebra  $A_2$  from  $A_1$  and so on. As a result, we acquire an ascending chain of  $L$ -algebras  $A = A_0 \subset A_1 \subset A_2 \subset \dots$ . Let  $\mathcal{A} = \bigcup_{k=0}^{\infty} A_k$ . In  $\mathcal{A}$ , every two nonzero elements generates the same ideal. Then  $\mathcal{A}$  is a simple  $L$ -algebra.  $\square$

**Theorem 5.10** *Every countably generated  $L$ -algebra over a countable field  $k$  can be embedded into a simple two-generated  $L$ -algebra.*

**Proof.** Let  $A$  be a countably generated  $L$ -algebra over a countable field  $k$ . We may assume that  $A$  has a countable  $k$ -basis  $X_0 = \{x_i | i \in \mathbb{N}^+\}$ . By Lemma 5.1,  $A = L(X_0|S_0)$ , where  $S_0 = \{x_i \prec x_j - \{x_i \prec x_j\}, x_i \succ x_j - \{x_i \succ x_j\} | i, j \in \mathbb{N}^+\}$ .

Let  $A_0 = L(X_0)$ ,  $A_0^+ = A_0 - \{0\}$  and fix the bijection

$$(A_0^+, A_0^+) \leftrightarrow \{(x_m^{(1)}, y_m^{(1)}) | m \in \mathbb{N}^+\}.$$

Let  $X_1 = X_0 \cup \{x_m^{(1)}, y_m^{(1)}, a, b | m \in \mathbb{N}^+\}$ ,  $A_1 = L(X_1)$ ,  $A_1^+ = A_1 - \{0\}$  and fix the bijection

$$(A_1^+, A_1^+) \leftrightarrow \{(x_m^{(2)}, y_m^{(2)}) | m \in \mathbb{N}^+\}.$$

For  $n \geq 1$ , let  $X_{n+1} = X_n \cup \{x_m^{(n+1)}, y_m^{(n+1)} | m \in \mathbb{N}^+\}$ ,  $A_{n+1} = L(X_{n+1})$ ,  $A_{n+1}^+ = A_{n+1} - \{0\}$  and fix the bijection

$$(A_{n+1}^+, A_{n+1}^+) \leftrightarrow \{(x_m^{(n+2)}, y_m^{(n+2)}) | m \in \mathbb{N}^+\}.$$

Consider the chain of the free  $L$ -algebras

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots.$$

Let  $X = \cup_{n=0}^{\infty} X_n$ . Then  $L(X) = \cup_{n=0}^{\infty} A_n$ .

Define

$$\begin{aligned} F_1 &= \{f_{1(ij)} = x_i \succ x_j - \{x_i \succ x_j\} | i, j \in \mathbb{N}^+\}, \\ F_2 &= \{f_{2(ij)} = x_i \prec x_j - \{x_i \prec x_j\} | i, j \in \mathbb{N}^+\}, \\ F_3 &= \{f_{3(i)} = a \prec \underbrace{(b \prec (\cdots (b \prec b)))}_i - x_i | i \in \mathbb{N}^+\}, \\ F_4 &= \{f_{4(lm)} = a \prec \underbrace{(b \prec \cdots \prec (b \prec (a \prec \cdots \prec (a \prec a))))}_m - x_m^{(l)} | m, l \in \mathbb{N}^+\}, \\ F_5 &= \{f_{5(lm)} = a \prec \underbrace{(b \prec \cdots \prec (b \succ (a \succ \cdots \succ (a \succ a))))}_m - y_m^{(l)} | m, l \in \mathbb{N}^+\}, \end{aligned}$$

Let  $F_{1-5} = F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5$ .

$$\begin{aligned} F_{6(1)} &= \{f_{6(1m)} = x_m^{(1)} \prec (f^{(0)} \succ y_m^{(1)}) - g^{(0)} \mid f^{(0)}, g^{(0)} \in k(\text{Irr}(F_1 \cup F_2) \cap L(X_0)), \\ &\quad f^{(0)} \text{ and } g^{(0)} \text{ are monic}\}, \\ F_{7(1)} &= \{f_{7(1ij)} = x_i \succ (((x_j \prec u) \prec v_1) \prec \cdots) \prec v_n - \\ &\quad (((\{x_i \succ x_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid u \in \text{Irr}(F_{1-5} \cup F_{6(1)}) \cap L(X_1) - X_0, \\ &\quad v_r \in \text{Irr}(F_{1-5} \cup F_{6(1)}) \cap L(X_1), r = 1, \dots, n, n \in \mathbb{N}\}, \end{aligned}$$

For  $l \geq 2$ , let

$$\begin{aligned} F_{6(l)} &= \{f_{6(lm)} = \underbrace{x_m^{(l)} \prec (\cdots \prec (x_m^{(l)} \prec (f^{(l-1)} \succ y_m^{(l)})))}_{|\overline{g^{(l-1)}}|} - g^{(l-1)} \mid f^{(l-1)}, g^{(l-1)} \in \\ &\quad k(\text{Irr}(F_{1-5} \cup F_{6(1)} \cup \cdots \cup F_{6(l-1)} \cup F_{7(1)} \cup \cdots \cup F_{7(l-1)}) \cap L(X_{l-1})) \text{ are monic}\}, \\ F_{7(l)} &= \{f_{7(lij)} = x_i \succ (((x_j \prec u) \prec v_1) \prec \cdots) \prec v_n - \\ &\quad (((\{x_i \succ x_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid n \in \mathbb{N}, \\ &\quad u \in \text{Irr}(F_{1-5} \cup F_{6(1)} \cup \cdots \cup F_{6(l)} \cup F_{7(1)} \cup \cdots \cup F_{7(l-1)}) \cap L(X_l) - X_0, \\ &\quad v_r \in \text{Irr}(F_{1-5} \cup F_{6(1)} \cup \cdots \cup F_{6(l)} \cup F_{7(1)} \cup \cdots \cup F_{7(l-1)}) \cap L(X_l), r = 1, \dots, n\}. \end{aligned}$$

Let  $F_6 = \cup_{l \geq 1} F_{6(l)}$ ,  $F_7 = \cup_{l \geq 1} F_{7(l)}$ .

Now let  $B = L(X|S)$ , where  $S = F_{1-5} \cup F_6 \cup F_7$ .

We prove that  $S$  is a Gröbner-Shirshov basis in  $L(X)$  with the ordering (2), where  $X$  is a well-ordered set.

The only ambiguity  $w$  of composition of inclusion in  $S$  is  $\overline{f_{7(lij)}}|_{\overline{f'_{7(i'j')}}}$  where  $\overline{f'_{7(i'j')}}$  is a subword of  $u$  or some  $v_r$ . It is trivial by a similar proof in Theorem 3.2.

The possible compositions of right multiplication are:  $f_{1(ij)} \prec u$  and  $f_{7(lij)} \prec u$ ,  $u \in N(X)$ ,  $l \in \mathbb{N}^+$ .

For  $u \in N(X)$ ,  $\exists l \in \mathbb{N}$  such that  $u \in L(X_l)$ . We can find an  $f = \sum_j \alpha_j w_j \in L(X_l)$ , where  $\alpha_j \in k$ ,  $w_j \in \text{Irr}(F_{1-5} \cup F_{6(1)} \cup \dots \cup F_{6(l)} \cup F_{7(1)} \cup \dots \cup F_{7(l-1)}) \cap L(X_l)$  such that  $u \equiv f \pmod{S}$ , and each  $w_j \leq u$ . So

$$f_{1(ij)} \prec u \equiv f_{1(ij)} \prec f \equiv \sum_j \alpha_j f_{1(ij)} \prec w_j \pmod{S}.$$

For each  $w_j$ , if  $w_j \notin X_0$ , then by  $F_{7(l)}$ ,

$$f_{1(ij)} \prec w_j = (x_i \succ x_j) \prec w_j - \{x_i \succ x_j\} \prec w_j \equiv 0 \pmod{S}.$$

If  $w_j = x_k \in X_0$ , then

$$f_{1(ij)} \prec w_j \equiv 0 \pmod{S}.$$

So all the compositions of right multiplication of  $f_{1(ij)}$  is trivial.

Now we consider the case of  $f_{7(lij)}$ . For any  $w \in N(X)$ ,  $\exists l' \in \mathbb{N}$  such that  $w \in L(X_{l'})$ . If  $l \geq l'$ , we can find an  $f = \sum_r \alpha_r w_r \in L(X_l)$ , where  $\alpha_r \in k$ ,  $w_r \in \text{Irr}(F_{1-5} \cup F_{6(1)} \cup \dots \cup F_{6(l)} \cup F_{7(1)} \cup \dots \cup F_{7(l-1)}) \cap L(X_l)$  such that  $w \equiv f \pmod{S}$ , and  $w_r \leq w$ . So

$$f_{7(lij)} \prec w \equiv f_{7(lij)} \prec f \equiv \sum_r \alpha_r f_{7(lij)} \prec w_r \equiv 0 \pmod{S}$$

since for each  $w_r$ ,  $f_{7(lij)} \prec w_r$  is of the form  $f_{7(lij)}$ .

Suppose that  $l < l'$ . For any  $t \in \{u, v_1, \dots, v_n, w\}$ , we can find an  $f = \sum_{tr} \alpha_{tr} w_{tr} \in L(X_{l'})$ , where  $\alpha_{tr} \in k$ ,  $w_{tr} \in \text{Irr}(F_{1-5} \cup F_{6(1)} \cup \dots \cup F_{6(l')} \cup F_{7(1)} \cup \dots \cup F_{7(l'-1)}) \cap L(X_{l'})$  such that  $t \equiv f \pmod{S}$ , and  $w_{tr} \leq t$ . Then  $f_{7(lij)} \prec w \equiv 0 \pmod{S}$ .

So  $S$  is a Gröbner-Shirshov basis in  $L(X)$ . By Theorem 4.6,  $A$  can be embedded into  $B = L(X|S)$ . By  $F_3$ ,  $F_4$  and  $F_5$ ,  $B$  is generated by  $\{a, b\}$ . Note that for every non-zero element  $f$  in  $B$ , there exists a monic polynomial  $f^{(l)} \in k(\text{Irr}(F_{1-5} \cup F_{6(1)} \cup \dots \cup F_{6(l)} \cup F_{7(1)} \cup \dots \cup F_{7(l)})) \cap L(X_l)$  such that  $\alpha f^{(l)} = f$  where  $\alpha \in k$ . Then by  $F_6$ , every two non-zero elements in  $B$  generate the same ideal. Thus,  $B$  is simple.

The proof is completed.  $\square$

**Theorem 5.11**  *$A, B, C$  are arbitrary  $L$ -algebras over a field  $k$ . If  $|k| \leq \dim(B * C)$  and  $|A| \leq |B * C|$ , where  $B * C$  is the free product of  $B$  and  $C$ , then  $A, B, C$  can be embedded into a simple  $L$ -algebra generated by  $B$  and  $C$ .*

**Proof.** We consider firstly the case that  $B$  and  $C$  are finite-dimensional. In this case,  $|k| \leq \aleph_0$  and  $|A| \leq \aleph_0$ .

We may assume that  $A$  has a countable  $k$ -basis  $X_A = \{a_i | i \in I_A\}$ ,  $B$  has a finite  $k$ -basis  $X_B = \{b_i | i \in I_B\}$  and  $C$  has a finite  $k$ -basis  $X_C = \{c_i | i \in I_C\}$ . By Lemma 5.1,  $A = L(X_A|S_A)$ , where  $S_A = \{a_i \prec a_j - \{a_i \prec a_j\}, a_i \succ a_j - \{a_i \succ a_j\} | i, j \in I_A\}$ ,  $B = L(X_B|S_B)$ , where  $S_B = \{b_i \prec b_j - \{b_i \prec b_j\}, b_i \succ b_j - \{b_i \succ b_j\} | i, j \in I_B\}$ ,  $C = L(X_C|S_C)$ , where  $S_C = \{c_i \prec c_j - \{c_i \prec c_j\}, c_i \succ c_j - \{c_i \succ c_j\} | i, j \in I_C\}$ .

Let  $X_0 = X_A \cup X_B \cup X_C$ ,  $L(X_0)^+ = L(X_0) - \{0\}$  and fix the bijection

$$(L(X_0)^+, L(X_0)^+) \leftrightarrow \{(x_m^{(1)}, y_m^{(1)}) | m \in \mathbb{N}^+\}.$$

For  $n \geq 0$ , let  $X_{n+1} = X_n \cup \{x_m^{(n+1)}, y_m^{(n+1)} | m \in \mathbb{N}^+\}$ ,  $L(X_{n+1})^+ = L(X_{n+1}) - \{0\}$  and fix the bijection

$$(L(X_{n+1})^+, L(X_{n+1})^+) \leftrightarrow \{(x_m^{(n+2)}, y_m^{(n+2)}) | m \in \mathbb{N}^+\}.$$

Let  $X = \cup_{n=0}^{\infty} X_n$ . Then  $L(X) = \cup_{n=0}^{\infty} L(X_n)$ .

Note that  $|X_A \cup \{x_m^{(n)}, y_m^{(n)} | m \in \mathbb{N}^+, n \in \mathbb{N}\}| = \aleph_0$ . Define

$$\begin{aligned} F_{1a} &= \{f_{1a(ij)} = a_i \succ a_j - \{a_i \succ a_j\} | i, j \in I_A\}, \\ F_{1b} &= \{f_{1b(ij)} = b_i \succ b_j - \{b_i \succ b_j\} | i, j \in I_B\}, \\ F_{1c} &= \{f_{1c(ij)} = c_i \succ c_j - \{c_i \succ c_j\} | i, j \in I_C\}, \\ F_{2a} &= \{f_{2a(ij)} = a_i \prec a_j - \{a_i \prec a_j\} | i, j \in I_A\}, \\ F_{2b} &= \{f_{2b(ij)} = b_i \prec b_j - \{b_i \prec b_j\} | i, j \in I_B\}, \\ F_{2c} &= \{f_{2c(ij)} = c_i \prec c_j - \{c_i \prec c_j\} | i, j \in I_C\}, \\ F_3 &= \{f_{3(i)} = \underbrace{(((b_0 \prec c_0) \prec (b_0 \prec c_0)) \prec \cdots \prec (b_0 \prec c_0))}_i \succ (b_0 \prec c_0) - d_i \mid i \in \mathbb{N}^+\}, \end{aligned}$$

where  $b_0 \in X_B$ ,  $c_0 \in X_C$  are two fixed elements and  $X_A \cup \{x_m^{(n)}, y_m^{(n)} | m \in \mathbb{N}^+, n \in \mathbb{N}\} = \{d_i | i \in \mathbb{N}^+\}$ .

Let  $F_{1-3} = F_{1a} \cup F_{1b} \cup F_{1c} \cup F_{2a} \cup F_{2b} \cup F_{2c} \cup F_3$ .

$$\begin{aligned} F_{4(1)} &= \{f_{4(1m)} = x_m^{(1)} \prec (f^{(0)} \succ y_m^{(1)}) - g^{(0)} \mid f^{(0)}, g^{(0)} \in \\ &\quad k(\text{Irr}(F_{1a} \cup F_{1b} \cup F_{1c} \cup F_{2a} \cup F_{2b} \cup F_{2c}) \cap L(X_0)) \text{ are monic}\}, \\ F_{5(a1)} &= \{f_{5(a1ij)} = a_i \succ (((a_j \prec u) \prec v_1) \prec \cdots) \prec v_n) - \\ &\quad (((\{a_i \succ a_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid u \in \text{Irr}(F_{1-3} \cup F_{4(1)}) \cap L(X_1) - X_A, \\ &\quad v_r \in \text{Irr}(F_{1-3} \cup F_{4(1)}) \cap L(X_1), r = 1, \dots, n, n \in \mathbb{N}\}, \\ F_{5(b1)} &= \{f_{5(b1ij)} = b_i \succ (((b_j \prec u) \prec v_1) \prec \cdots) \prec v_n) - \\ &\quad (((\{b_i \succ b_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid u \in \text{Irr}(F_{1-3} \cup F_{4(1)}) \cap L(X_1) - X_B, \\ &\quad v_r \in \text{Irr}(F_{1-3} \cup F_{4(1)}) \cap L(X_1), r = 1, \dots, n, n \in \mathbb{N}\}, \\ F_{5(c1)} &= \{f_{5(c1ij)} = c_i \succ (((c_j \prec u) \prec v_1) \prec \cdots) \prec v_n) - \\ &\quad (((\{c_i \succ c_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid u \in \text{Irr}(F_{1-3} \cup F_{4(1)}) \cap L(X_1) - X_C, \\ &\quad v_r \in \text{Irr}(F_{1-3} \cup F_{4(1)}) \cap L(X_1), r = 1, \dots, n, n \in \mathbb{N}\}, \\ F_{5(31)} &= \{f_{5(31i)} = \underbrace{(((b_0 \prec c_0) \prec (b_0 \prec c_0)) \prec \cdots \prec (b_0 \prec c_0))}_i \\ &\quad \succ (((((b_0 \prec c_0) \prec u) \prec v_1) \prec \cdots) \prec v_n) - (((d_i \prec u) \prec v_1) \prec \cdots) \prec v_n \mid \\ &\quad u, v_r \in \text{Irr}(F_{1-3} \cup F_{4(1)}) \cap L(X_1), r = 1, \dots, n, n \in \mathbb{N}\}. \end{aligned}$$

Denote  $T_l = F_{1-3} \cup F_{4(1)} \cup \cdots \cup F_{4(l)} \cup F_{5(a1)} \cup \cdots \cup F_{5(al)} \cup F_{5(b1)} \cup \cdots \cup F_{5(bl)} \cup F_{5(c1)} \cup \cdots \cup F_{5(cl)} \cup F_{5(31)} \cup \cdots \cup F_{5(3l)}$ . Then  $T_1$  is defined. We use induction to define  $T_l$ .

Assume  $l \geq 2$ . Let

$$\begin{aligned}
F_{4(l)} &= \{f_{4(lm)} = \underbrace{x_m^{(l)} \prec (\cdots \prec (x_m^{(l)} \prec (f^{(l-1)} \succ y_m^{(l)}))}_{|g^{(l-1)}|} - g^{(l-1)} \mid \\
&\quad f^{(l-1)}, g^{(l-1)} \in k(\text{Irr}(T_{l-1}) \cap L(X_{l-1})) \text{ are monic}\}, \\
F_{5(al)} &= \{f_{5(alij)} = a_i \succ (((a_j \prec u) \prec v_1) \prec \cdots) \prec v_n - \\
&\quad (((\{a_i \succ a_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid u \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l) - X_A, \\
&\quad v_r \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l), r = 1, \dots, n, n \in \mathbb{N}\}, \\
F_{5(bl)} &= \{f_{5(bl ij)} = b_i \succ (((b_j \prec u) \prec v_1) \prec \cdots) \prec v_n - \\
&\quad (((\{b_i \succ b_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid u \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l) - X_B, \\
&\quad v_r \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l), r = 1, \dots, n, n \in \mathbb{N}\}, \\
F_{5(cl)} &= \{f_{5(cl ij)} = c_i \succ (((c_j \prec u) \prec v_1) \prec \cdots) \prec v_n - \\
&\quad (((\{c_i \succ c_j\} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid u \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l) - X_C, \\
&\quad v_r \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l), r = 1, \dots, n, n \in \mathbb{N}\}, \\
F_{5(3l)} &= \{f_{5(3li)} = \underbrace{(((b_0 \prec c_0) \prec (b_0 \prec c_0)) \prec \cdots \prec (b_0 \prec c_0))}_i \\
&\quad \succ (((((b_0 \prec c_0) \prec u) \prec v_1) \prec \cdots) \prec v_n) - (((d_i \prec u) \prec v_1) \prec \cdots) \prec v_n \mid \\
&\quad u, v_r \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l), r = 1, \dots, n, n \in \mathbb{N}\}.
\end{aligned}$$

Let  $F_4 = \cup_{l \geq 1} F_{4(l)}$ ,  $F_5 = \cup_{i=a,b,c,3} (\cup_{l \geq 1} F_{5(il)})$ . Now let  $D = L(X|S)$  where  $S = F_{1-3} \cup F_4 \cup F_5$ .

By a similar proof in Theorem 5.10,  $S$  is a Gröbner-Shirshov basis in  $L(X)$  with the ordering (2), where  $X$  is a well-ordered set. By Theorem 4.6,  $A$ ,  $B$  and  $C$  can be embedded into  $D = L(X|S)$ . By  $F_3$ ,  $D$  is generated by  $B$  and  $C$ . By  $F_4$ ,  $D$  is simple. Now, we finish the proof in the case of finite-dimensional  $L$ -algebras  $B$  and  $C$ .

Consider secondly the case that one of  $B$  and  $C$  is infinite dimensional. Assume that  $\dim B \leq \dim C$ ,  $\dim C = \alpha \geq \aleph_0$ . Since  $|k| \leq \dim(B * C)$ , so  $\dim(B * C) = |B * C| = \alpha$ . Without loss of generality, we can assume that  $\dim A = \alpha$  and so  $|A| = \alpha$ .

In this case, we only need to change  $F_3$  into

$$\begin{aligned}
F_3 &= \{f_{3(i\beta)} = \underbrace{(((b_0 \prec c_0) \prec (b_0 \prec c_0)) \prec \cdots \prec (b_0 \prec c_0))}_i \succ (b_0 \prec c_\beta) - d_{i\beta} \mid \\
&\quad i \geq 1, 1 \leq \beta < \alpha\},
\end{aligned}$$

where  $X_A \cup \{x_m^{(n)}, y_m^{(n)} \mid 1 \leq m < \alpha, n \geq 1\} = \{d_{i\beta} \mid i \geq 1, 1 \leq \beta < \alpha\}$  with cardinal number  $\alpha$ , and  $\{c_\beta \mid 1 \leq \beta < \alpha\} \subseteq X_C$  such that there is a one-to-one correspondence  $\{(i, c_\beta) \mid i \in \mathbb{N}^+, 1 \leq \beta < \alpha\} \leftrightarrow \{d_{i\beta} \mid i \in \mathbb{N}^+, 1 \leq \beta < \alpha\}$ .

Now, in order to keep  $S$  as a Gröbner-Shirshov basis, we change  $F_{5(3l)}$ ,  $l \in \mathbb{N}^+$  into

$$\begin{aligned}
F_{5(3l)} &= \{f_{5(3li\beta)} = \underbrace{(((b_0 \prec c_0) \prec (b_0 \prec c_0)) \prec \cdots \prec (b_0 \prec c_0))}_i \\
&\quad \succ (((((b_0 \prec c_\beta) \prec u) \prec v_1) \prec \cdots) \prec v_n) - (((d_{i\beta} \prec u) \prec v_1) \prec \cdots) \prec v_n \mid \\
&\quad u, v_r \in \text{Irr}(T_{l-1} \cup F_{4(l)}) \cap L(X_l), r = 1, \dots, n, n \in \mathbb{N}, i \geq 1, 1 \leq \beta < \alpha\}.
\end{aligned}$$

Then by the same analysis,  $A$ ,  $B$  and  $C$  can be embedded into  $D = L(X|S)$  which is a simple  $L$ -algebra generated by  $B$  and  $C$ .

The proof is completed. □

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